GOVERNMENT OF ANDHRA PRADESH
COMMISSIONERATE OF COLLEGIATE EDUCATION

INNER PRODUCTS

PARALLELOGRAM LAW

MATHEMATICS

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Learning Objectives

Students will be able to recognize the following properties of a parallelogram:

• Describe the relationships between opposite sides in a parallelogram.

i.e Opposite sides are parallel and Opposite sides are equal in length.

• Describe the relationship between opposite angles in a parallelogram.

i.e Opposite sides are equal in measure.

• Describe the relationship between consecutive angles in a parallelogram.

The sum of two adjacent angles is 180^0.

• Describe the relationship between the two diagonals in a parallelogram.

i.e The diagonals bisect each other.

In mathematics, the simplest form of the **parallelogram law** (also called the **parallelogram identity**) belongs to elementary geometry.

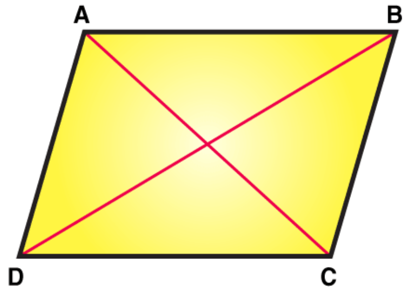
It states that the sum of the squares of the lengths of the four sides of a parallelogram equals the sum of the squares of the lengths of the two diagonals.

We use these notations for the sides: *AB*, *BC*, *CD*, *DA*.

But since in Euclidean geometry a parallelogram necessarily has opposite sides equal, that is, $AB = CD$ and $BC = DA$,

the law can be stated as

$2\,AB^2 + 2\,BC^2 = AC^2 + BD^2$



If the parallelogram is a rectangle, the two diagonals are of equal lengths $AC = BD$,

so $2\,AB^2 + 2\,BC^2 = 2\,AC^2$ and the statement reduces to the Pythagorean theorem.

Parallelogram Law Proof

Let $AD = BC = x$, $AB = DC = y$, and $\angle BAD = \alpha$

Using the law of cosines in the triangle BAD, we get

$x^2 + y^2 - 2xy\cos(\alpha) = BD^2$ ———-(1)

We know that in a parallelogram, the adjacent angles are supplementary.

So $\angle ADC = 180 - \alpha$

Now, again use the law of cosines in the triangle ADC

$x^2 + y^2 - 2xy\cos(180 - \alpha) = AC^2$ ———-(2)

Apply trigonometric identity $\cos(180 - x) = -\cos x$ in (2)

$x^2 + y^2 + 2xy\cos(\alpha) = AC^2$

Now, the sum of the squares of the diagonals ($BD^2 + AC^2$) are represented as,

$BD^2 + AC^2 = x^2 + y^2 – 2xy\cos(\alpha) + x^2 + y^2 + 2xy \cos(\alpha)$

Simplify the above expression, we get;

$BD^2 + AC^2 = 2x^2 + 2 y^2$ ———-(3)

The above equation is represented as:

$BD^2 + AC^2 = 2(AB)^2 + 2(BC)^2$

Hence, the parallelogram law is proved.

The Parallelogram Identity for Inner Product Spaces

If V is an inner product space and $\alpha, \beta \in V$ then the sum of squares of the norms of the vectors $\alpha+\beta$ and $\alpha-\beta$ equals twice the sum of the squares of the norms of the vectors α and β , that is: $||\alpha + \beta||^2 + ||\alpha - \beta||^2 = 2(||\alpha||^2 + ||\beta||^2)$

Statement: Let V be an inner product space and let $\alpha , \beta \in V$. Then,

$$||\alpha + \beta||^2 + ||\alpha - \beta||^2 = 2(||\alpha||^2 + ||\beta||^2)$$

Proof:

$$||\alpha + \beta||^2 = \langle\alpha + \beta, \alpha + \beta\rangle = \langle\alpha, \alpha\rangle + \langle\alpha, \beta\rangle + \langle\beta, \alpha\rangle + \langle\beta, \beta\rangle$$

$$= ||\alpha||^2 + ||\beta||^2 + \langle\alpha, \beta\rangle + \langle\beta, \alpha\rangle \quad \ldots\ldots.(1)$$

and $||\alpha - \beta||^2 = \langle \alpha - \beta, \alpha - \beta \rangle = \langle \alpha, \alpha \rangle + \langle \alpha, -\beta \rangle + \langle -\beta, \alpha \rangle + \langle -\beta, -\beta \rangle$

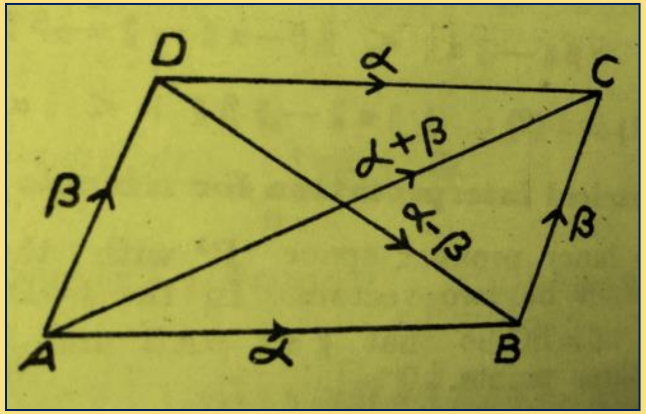
$= ||\alpha||^2 + ||\beta||^2 - \langle \alpha, \beta \rangle - \langle \beta, \alpha \rangle$ ……..(2)

Adding (1) and (2) we get

$||\alpha + \beta||^2 + ||\alpha - \beta||^2 = 2(||\alpha||^2 + ||\beta||^2)$

Geometrical interpretation of Parallelogram law

Let α, β be two vectors in the inner product space R^3 with standard inner product. In 3-D space R^3 we can take $\alpha = \overline{AB}, \beta = \overline{BC}$ So that ||α|| = AB , ||β|| = BC where A, B, C, D are points . Then the vectors α + β, α - β represents the diagonals $\overline{AC}, \overline{DB}$ of the parallelogram ABCD.



So, || α + β || = AC and || α - β || = DB.

From the parallelogram law theorem,

$$\| \alpha + \beta \|^2 + \| \alpha - \beta \|^2 = 2(\|\alpha\|^2 + \|\beta\|^2)$$

$$\Rightarrow AC^2 + DB^2 = 2 (AB^2 + BC^2)$$

$$= AB^2 + BC^2 + CD^2 + DA^2$$

⇒The sum of the squares on the diagonals of a parallelogram is equal to the sum of the squares on the four sides.


Thank you